

Normality and Finite-State Dimension of Liouville Numbers

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Abstract. Liouville numbers were the first class of real numbers which were proven to be transcendental. It is easy to construct non-normal Liouville numbers. Kano [13] and Bugeaud [4] have proved, using analytic techniques, that there are normal Liouville numbers. Here, for a given base $k \geq 2$, we give two simple constructions of a Liouville number which is normal to the base k .

The first construction is combinatorial, and is based on de Bruijn sequences. A real number in the unit interval is normal if and only if its finite-state dimension is 1. We generalize our construction to prove that for any rational r in the closed unit interval, there is a Liouville number with finite state dimension r . This refines Staiger's result [19] that the set of Liouville numbers has constructive Hausdorff dimension zero, showing a new quantitative classification of Liouville numbers can be attained using finite-state dimension.

In the second number-theoretic construction, we use an arithmetic property of numbers - the existence of primitive roots - to construct Liouville normal numbers. We show that if Artin's conjecture on primitive roots holds for a number a , it is possible to construct Liouville numbers normal in base a .

1 Introduction

One of the important open questions in the study of normality is whether any algebraic irrational number is normal. On the other hand, it is known that there are normal transcendentals as well as non-normal transcendentals. For example, Mahler has proved that the Champernowne constant [16] as well as the Thue-Morse [15] constant are transcendental. However, the Champernowne constant is normal [5], whereas the Thue-Morse constant has finite-state dimension 0 [2].

Liouville numbers were the first class of numbers which were proven transcendental. We want to know whether there are normal Liouville numbers. This question cannot be settled by a counting or a measure argument: the set of Normal numbers has measure 1, while the set of Liouville numbers has measure 0 (indeed, Hausdorff dimension 0). This means that none of the scenarios - there being no normal Liouville number, every Liouville number being normal, and

there being a partial overlap between the set of normal numbers and the set of Liouville numbers - can be ruled out [17]. Thus there is no easy existential argument based on measure-theory alone.

In this paper, we show that there are Liouville numbers which are non-normal, and others which are normal. Indeed, there are Liouville numbers of every rational finite-state dimension between 0 and 1 (definitions follow in Section 3).

Examples of non-normal Liouville numbers are well-known. Normal Liouville numbers are harder to construct, and there are works by Kano [13] and Bugeaud [4] which establish the existence of such numbers. Kano constructs, for any bases a and b , Liouville numbers which are normal in base a but not in base ab . Bugeaud gives a non-constructive proof using Fourier analytic techniques that there are Liouville numbers which are absolutely normal – that is, normal in all bases. In this paper, we give a combinatorial construction of a number that is normal to a given base b . The construction is elementary. Thus the Liouville numbers forms a class of numbers whose transcendence is easy to establish, and which contain simple examples of normal and non-normal numbers.

The set of normal numbers coincide exactly with the set of numbers with finite-state dimension 1 [18], [6]. We show that the combinatorial nature of our construction lends itself to the construction of Liouville numbers of any finite-state dimension. Thus we get a quantitative classification of non-normal Liouville numbers. This classification is new, since the set of Liouville numbers has classical Hausdorff dimension and even effective Hausdorff Dimension zero [19].

In the concluding section of the paper, we give a conditional construction of a normal Liouville number. We show that the notion of a primitive root of a prime can be utilized to construct normal numbers. Artin’s conjecture [1] says that for any integer a that is not -1 and not a perfect square, there are infinitely many primes for which a is a primitive root. We show how this can be utilized in the construction of a normal number. This connection illuminates how arithmetic properties of natural numbers have a bearing on properties of real numbers. Moreover, this construction can be viewed as a first step towards an explicit construction of an absolutely normal number.

We begin with a survey briefly explaining Liouville’s approximation theorem and defining the class of Liouville numbers. Section 3 constructs a Liouville number which is disjunctive – that is to say, has all strings appearing in its base b expansion – but is still not normal. The subsequent section gives the construction of a normal Liouville number.

2 Liouville’s Constant and Liouville Numbers

Liouville’s approximation theorem says that algebraic irrationals are inapproximable by rational numbers to arbitrary precisions.

Theorem 1 (Liouville’s Theorem). *Let β be a root of $f(x) = \sum_{j=0}^n a_j x^j \in \mathbb{Z}[x]$. Then there is a constant C_β such that for every pair of integers a and b , $b > 0$, we have $|\beta - \frac{a}{b}| > \frac{C_\beta}{b^n}$.*

Liouville then constructed the following provably irrational number $\psi_1 = \sum_{i=0}^{\infty} 10^{-i!}$, and showed that it had arbitrarily good rational approximations in the above sense, and therefore is a transcendental number.

Definition 1. A real number α in the unit interval is called a Liouville number if for all numbers n , there are numbers $p > 0$ and $q > 1$ such that $\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^n}$.

For every n , we have

$$\left| \psi_1 - \sum_{i=1}^n \frac{1}{2^{i!}} \right| = \sum_{i=n+1}^{\infty} \frac{1}{2^{i!}} = \sum_{i=(n+1)!}^{\infty} \frac{1}{2^i} = \frac{1}{2^{(n+1)!-1}} < \frac{1}{2^{n!n}} = \frac{1}{q_n^n},$$

where q_n is the denominator of the finite sum $\sum_{i=1}^n \frac{1}{2^{i!}}$. Thus ψ_1 is a Liouville number.

It is easy to see that the Liouville constant ψ_1 is not a normal number - the sequence 111 never appears in the decimal expansion of ψ_1 . It is natural to investigate whether all Liouville numbers are non-normal.

This requires sharper observations than the one above.

3 Disjunctive Liouville Sequences

Hertling [12] has showed that there are disjunctive Liouville numbers - that is, there are Liouville numbers whose base r expansions have all possible r -alphabet strings. Staiger strengthened this result to show that there are Liouville numbers which are disjunctive in any base [19]. This shows that we cannot rely on the above argument of absent strings to show non-normality.

Here, to motivate the construction of a normal number in the next section, we give a different construction of a different disjunctive Liouville sequence. Consider $\psi_2 = \sum_{i=3}^{\infty} \frac{i}{2^{i!}}$. For any binary string w , we know that $1w$ (1 concatenated with w) is the binary representation of a number, hence it appears in the binary expansion of ψ_2 . Thus ψ_2 is a disjunctive sequence.

It is easy to see that ψ_2 is not a normal number. At all large enough prefix lengths of the form $n!$, there are at most $n(\lfloor \log_2 n \rfloor + 1)$ ones - this follows from the fact that at most n unique non-zero numbers have appeared in the binary expansion of ψ_2 , and each of the numbers can be represented with at most $\lfloor \log_2 n \rfloor + 1$ bits. Hence

$$\liminf_{n \rightarrow \infty} \frac{|\{i : 0 \leq i \leq n-1 \text{ and } \psi_2[i] = 1\}|}{n} \leq \lim_{n \rightarrow \infty} \frac{n(\lfloor \log_2 n \rfloor + 1)}{n!} = 0,$$

which proves that ψ_2 is not normal.

However, ψ_2 is a Liouville number: For every n , there are rationals with denominators of size $2^{n!}$ which satisfy the Liouville criterion, as follows:

$$\left| \psi - \sum_{i=3}^n \frac{i}{2^{i!}} \right| = \sum_{i=n+1}^{\infty} \frac{i}{2^{i!}} < \sum_{k=1}^{\infty} \frac{n+k}{2^{(n+1)! \cdot k}}$$

Summing up the series, we obtain the inequality

$$\sum_{k=1}^{\infty} \frac{n+k}{2^{(n+1)! \cdot k}} = \frac{[n+1] \cdot 2^{(n+1)!-1} + 1}{2^{((n+1)!-1) \cdot 2}} < \frac{[n+2]}{2^{(n+1)!-1}} < \frac{1}{2^{(n!) \cdot n}}.$$

4 A Normal Liouville Number

Though the Liouville numbers constructed above were non-normal, there are normal Liouville numbers. We give such a construction below, which depends on DE BRUIJN SEQUENCES introduced by de Bruijn [7] and Good [10], a standard tool in the study of normality.

Definition 2. Let Σ be an alphabet with size k . A k -ary de Bruijn sequence $B(k, n)$ of order n , is a finite string for which every possible string in Σ^n appears exactly once, counting in a circular fashion. That is, for every $w \in \Sigma^n$,

$$|\{0 \leq i \leq n-1 \mid B(k, n)[i \dots (i+n) \bmod k^n] = w\}| = 1.$$

de Bruijn proved that such sequences exist for all k and all orders n . Since each de Bruijn sequence $B(k, n)$ contains each n -length string exactly once, it follows that the length of $B(k, n)$ is exactly k^n .

4.1 An Initial Attempt

Since the de Bruijn sequence $B(2, n)$ have equal number of occurrences of all strings with fewer than n bits (counting in a circular manner), it is natural to attempt constructions of normal numbers by extending de Bruijn numbers: for example, consider a number whose binary expansion starts with $B(2, 1)$. Then we look for de Bruijn sequence of order 2^1 , circularly shifting it to start with $B(2, 1)$. Thus, we can hope to achieve equal number of occurrences for any binary string w .

However, to use this approach, we have to calculate the *discrepancy* of the de Bruijn sequences thus considered - We have to ensure that the discrepancy of the constructed number is fairly low: For every $\epsilon > 0$, for all but finitely many prefixes, the frequency of the occurrence of a given string w is bounded within $2^{-|w|} \pm \epsilon$. This involves analytic bounds like the Erdős-Turán bound [8] [9] (see for example, [14]), which we wish to avoid.

Another difficulty is to simultaneously ensure that the number thus constructed is a Liouville number. Thus extensions of de Bruijn sequences to form seem inadequate for the construction of a normal Liouville number. We will construct such a number by *repeating* de Bruijn sequences.

4.2 Construction

If w is any string, we write w^i for the string formed by repeating w , i times. In this section, we limit ourselves to the binary alphabet $\Sigma = \{0, 1\}$ even though the construction generalizes to all alphabets.

Consider $\alpha \in [0, 1)$ with binary expansion defined as follows.

$$\alpha = 0 . B(2, 1)^{1^1} B(2, 2)^{2^2} B(2, 3)^{3^3} \dots B(2, i)^{i^i} \dots$$

Informally, we can explain why this construction defines a normal Liouville number, as follows. The Liouville numbers ψ_1 and ψ_2 that we considered before, have prefixes that are mostly zeroes. The density of 1s go asymptotically to zero as we consider longer and longer prefixes. So it is fairly easy for a finite state compressor to compress the data in a prefix. However, in this construction, the repeating patterns employed are those which are eventually hard for any given finite state compressor. This is why the sequence could be normal.

Moreover, the transition in the patterns occur at prefix lengths of the form k^k . By Stirling's approximation,

$$k! \cong k^k e^{-k} \sqrt{2\pi k}.$$

So the transitions in the pattern occur at prefix lengths similar to that of ψ_1 and ψ_2 , so it is reasonable to expect a sequence of rationals approximating α that obeys the Liouville criterion. We now make this argument more precise.

For any i , let $n_i = \sum_{m=1}^i m^m 2^m$. We call the part of $\alpha[n_{i-1} \dots n_i - 1]$ as the i^{th} stage of α , which consists of i^i copies of $B(2, i)$. Thus n_i denotes the length of the prefix of α which has been defined at the end of the i^{th} stage. We have the following estimate for n_i .

$$n_i = \sum_{m=1}^i m^m 2^m < i^i \sum_{m=1}^i 2^m = i^i [2^{i+1} - 2] = i^i 2^{i+1} - 2i^i < 2(i^i 2^i).$$

Thus $n_i = O([2i]^i)$.

Lemma 1. α is a Liouville number.

Proof. Consider the rational number $\frac{p_i}{q_i}$ ³ with a binary expansion which coincides with α until the $i - 1^{\text{st}}$ stage, followed by a recurring block of $B(2, i)$. This rational number is

$$\frac{\alpha[0 \dots n_{i-1}]}{2^{n_{i-1}}} + \frac{B(2, i)}{(2^{2^i} - 1)2^{n_{i-1}}},$$

obtained by evaluating the binary expansion as a geometric series. The *exponent* of the denominator of this rational number is

$$2^i + n_{i-1} = 2^i + O([2(i-1)]^{i-1}) = O(n_{i-1}),$$

so the denominator of the rational is $2^{O(n_{i-1})}$.

We add i^i copies of $B(2, i)$ in the i^{th} stage. Thus the expansion of α and that of r_i coincide for the first n_i positions. So, α and $\frac{p_i}{q_i}$ are in the same dyadic interval of length at most 2^{-n_i} , and hence are within $\frac{1}{2^{n_i}}$ of each other.

³ not necessarily in the lowest form

We have also that $n_i > iO([2(i-1)]^{(i-1)})$, so that

$$\frac{1}{2^{n_i}} < \frac{1}{(2^{O(n_{i-1})})^i}.$$

Thus, $\left| \alpha - \frac{p_i}{q_i} \right| < \frac{1}{q_i^i}$. Since this is true of any stage i , we can see that α is a Liouville number. \square

Lemma 2. α is normal to the base 2.

Proof. Let us define $\text{COUNT} : \Sigma^* \times \Sigma^* \rightarrow \mathbb{N}$ by

$$\text{COUNT}(w; x) = |\{n \mid x[n \dots n + |w| - 1] = w\}|,$$

that is, the number of times w occurs in x , counting in a sliding block fashion. For example, 00 occurs twice in 1000. It is enough to show that for an arbitrary binary string w of length m , for all large enough indices j ,

$$\text{COUNT}(w; \alpha[0 \dots j - 1]) = 2^{-m}j + o(j).$$

Let j be a number greater than n_m . Every such index j has a number i such that $n_i < j \leq n_{i+1}$.

We split the analysis into three phases, that of the prefix $\alpha[0 \dots n_{m-1} - 1]$, of the middle region $\alpha[n_{m-1} \dots n_i]$, and of the suffix $\alpha[n_i \dots j - 1]$.

The prefix $\alpha[0 \dots n_{m-1}]$ has a constant length that depends on w but not on j . Hence the discrepancy in the count of w due to this prefix, which is at most n_{m-1} , is $o(j)$.

Since the number of times w occurs in $B(2, m)$ is exactly 1,

$$\text{COUNT}(w; B(2, m)) = 2^{-m} |B(2, m)|.$$

Similarly, it is easy to see that for any $M > m$, by the properties of the de Bruijn sequences,

$$\text{COUNT}(w; B(2, M)) = 2^{-m} |B(2, M)|.$$

This observation is used in the following analysis of the middle part and the suffix.

The part of α in the stretch $n_{m-1} \dots n_m - 1$ parses exactly into m^m disjoint blocks of $B(2, m)$. Consequently,

$$\text{COUNT}(w; \alpha[n_{m-1} \dots n_m]) = 2^{-m}(n_m - n_{m-1}).$$

For all stages k between $m - 1$ and i ,

$$\text{COUNT}(w; \alpha[n_k \dots n_{k+1} - 1]) = 2^{-m}(n_{k+1} - n_k),$$

hence by a telescoping sum,

$$\text{COUNT}(w; \alpha[n_{m-1} \dots n_i - 1]) = 2^{-m}(n_i - n_{m-1}).$$

The suffix is formed during the $i + 1^{\text{st}}$ stage of construction of α , and hence consists of $(i + 1)^{(i+1)}$ copies of $B(2, i + 1)$. Let j be such that

$$p \cdot 2^{(i+1)} < j - n_i < (p + 1)2^{i+1} - 1.$$

That is, j falls within the $p + 1^{\text{st}}$ copy of $B(2, i + 1)$. Then,

$$j = n_i + p2^{i+1} + o(j),$$

since the last term is at most $2^{i+1} - 1$ and $n_i = \Omega(i^i)$.

Since w is normally distributed in $\alpha[0 \dots n_i]$ and in each of the p preceding copies of $B(2, i + 1)$, we have

$$\text{COUNT}(w; \alpha[0 \dots j - 1]) = 2^{-m}n_i + 2^{-m}p2^{i+1} + o(j) = 2^{-m}j + o(j),$$

showing that α is normal. \square

5 Finite State Dimension

We now briefly give the block entropy characterization of finite state dimension [3]. Finite-state dimension, or equivalently finite state compressibility, is an asymptotic measure of information density in a sequence measured by a finite-state automaton. This was introduced by Dai, Lathrop, Lutz and Mayordomo [6].

Finite-state dimension provides a classification of sequences based on their information density. The sequences with maximal density, are exactly the set of normal sequences. These have finite-state dimension 1.

Let Ω be a nonempty finite set. Recall that the *Shannon entropy* of a probability measure π on Ω is

$$H(\pi) = \sum_{w \in \Omega} \pi(w) \log \frac{1}{\pi(w)},$$

where $0 \log \frac{1}{0} = 0$.

For nonempty strings $w, x \in \Sigma^+$, we write

$$\#(w, x) = \left| \left\{ m \leq \frac{|x|}{|w|} - 1 \mid w = x[m|w| \dots (m+1)|w| - 1] \right\} \right|.$$

That is, $\#(w, x)$ is the number of times a string w of length m occurs in x , when x is parsed into disjoint blocks each of length m .

For each infinite binary sequence and positive integer n , and a binary string w of length m , the n^{th} block frequency of w in S is

$$\pi_{S,n}(w) = \frac{\pi(w, S[0 \dots n|w| - 1])}{n}.$$

This defines a probability measure on m long binary strings. The normalized lower block entropy rates of S is

$$H_l^-(S) = \frac{1}{l} \liminf_{n \rightarrow \infty} H \left(\pi_{S,n}^{(l)} \right)$$

Definition 3. Let $S \in \Sigma^\infty$. The finite state dimension of S is

$$\dim_{FS}(S) = \inf_{l \in \mathbb{Z}^+} H_l^-(S).$$

For purposes of the next section, we use a sliding block variant of the block entropy. This is obtained by counting the frequency of blocks in a sliding block fashion. Let $0 < m < n$ be integers. The frequency of an m -long block w in an n -long string x is defined as

$$\frac{|\{i \mid w = x[i \dots i + m - 1]\}|}{n - m + 1}.$$

It is easy to verify that this defines a probability measure over the set of m -long strings, and hence it is possible to define the sliding block entropy in a manner analogous to the definition of the block entropy.

It is implicit in the work of Ziv and Lempel [20] that the sliding block entropy and the block entropy are both equal to the finite state compressibility of a sequence. In the following section, we establish that it is possible to attain every rational sliding block entropy value using Liouville numbers.

6 Finite State Dimension of Liouville Numbers

We now have β , a Liouville number with finite-state dimension zero, and α , a normal Liouville number - that is, a number with finite-state dimension 1. We show, that for any rational $q \in [0, 1]$, we can construct a Liouville number having finite-state dimension q . The construction is a variant of the standard dilution argument in finite-state dimension [6].

The dilution argument is as follows. Suppose S is an infinite binary sequence, and $\frac{p}{q}$ is a rational in the unit interval expressed in lowest terms. Then,

$$w_0 0^{q-p} w_1 0^{q-p} \dots,$$

where w_0 is the first p bits of S , w_1 is the next p bits of S , and so on, is a binary sequence with finite-state dimension $\frac{p}{q} \dim_{FS}(S)$. We cannot adopt this construction, since it is not certain that a dilution of α gives us a Liouville number even if it gives a sequence with the desired finite-state dimension.

The Liouville construction depends crucially on the fact that the i^{th} stage is the repetition of a block. This enables us to design a rational number sequence converging to α while satisfying the Liouville criterion. The standard dilution may not have periodic stretches which suit approximation in the Liouville sense.

We show that a slight variant of the dilution argument enables us to create a Liouville number with arbitrary rational finite-state dimension.

Let us establish that we can construct a Liouville number with finite state dimension $\frac{m}{n}$, where m and n are positive numbers, and the rational is expressed in lowest terms.

The sequence we construct is

$$\alpha_{m/n} = 0 \cdot \left((0^{2^1})^{(n-m)} B(2, 1)^m \right)^{1^1} \dots \left((0^{2^k})^{(n-m)} B(2, k)^m \right)^{k^k} \dots$$

That is, the recurring block in the k th stage consists of m copies of $B(2, k)$ and a padding of $n - m$ copies of 0^{2^k} . The recurring block has a length of $n2^k$. This block is then repeated k^k times, to form the k th stage. This is similar to what happens in the construction of the normal Liouville number.

We now show that the constructed number has the desired finite-state dimension.

First, we count the frequency of k -long strings in the stage k . Any string other than 0^k occurs m times, and 0^k occurs $(n - m)2^k + m$ times.

Thus the frequency in the block, of any string other than 0^k is $\frac{m}{n2^k}$, and the frequency of 0^k is $\frac{n-m}{n} + \frac{m}{n2^k}$.

We now compute the k -block entropy of the k th stage. This is

$$\frac{1}{k} \left[\frac{m}{n2^k} (2^k - 1) \log \frac{n2^k}{m} + \left(\frac{n-m}{n} + \frac{m}{n2^k} \right) \log \frac{n2^k}{(n-m)2^k + m} \right] \quad (1)$$

$$= \frac{1}{k} \left[\frac{m}{n2^k} 2^k \log(n2^k) - \frac{m}{n2^k} (2^k - 1) \log m \right. \\ \left. - \frac{m}{n2^k} \log((n-m)2^k + m) + \frac{n-m}{n2^k} \log \frac{n2^k}{(n-m)2^k + m} \right] \quad (2)$$

$$= \frac{m}{n} \frac{\log(n2^k)}{k} - \frac{m}{n2^k} (2^k - 1) \frac{\log m}{k} \\ - \frac{m}{n2^k} \frac{\log((n-m)2^k + m)}{k} + \frac{n-m}{n2^k} \frac{1}{k} \log \frac{n2^k}{(n-m)2^k + m} \quad (3)$$

We now simplify the terms to get the following expression for the k -block entropy of the k^{th} stage.

$$\frac{m}{n} \Theta(1) - \frac{m}{n} \left(1 - \Theta\left(\frac{1}{2^k}\right) \right) \Theta\left(\frac{1}{k}\right) - \frac{m}{n2^k} \frac{\Theta(k)}{k} + \frac{n-m}{n2^k} \frac{1}{k} \log \frac{n2^k}{(n-m)2^k + m}. \quad (4)$$

The last term can be bounded using the following analysis. We know, since $0 < m < n$, that $n2^k > (n-m)2^k \geq 2^k$. Hence,

$$2^k < m + 2^k < m + (n-m)2^k < m + n2^k < 2n2^k,$$

the extreme terms being obtained by the bounds $0 < m < n$.

Hence,

$$\log \frac{n2^k}{2^k} > \log \frac{n2^k}{m + (n-m)2^k} > \log \frac{n2^k}{2n2^k},$$

hence

$$\log n > \log \frac{n2^k}{m + (n-m)2^k} > \log \frac{1}{2}.$$

Since both the upper bound and lower bounds are constants independent of k , we have that the last term in (4) is $\Theta(1/2^k)$.

The same bound holds for k -block entropy of any stage $K > k$, by the property of $B(2, K)$.

Thus, taking limits as $k \rightarrow \infty$, the sliding block entropy rate of the sequence is $\frac{m}{n}$, as desired.

We now show that $\alpha_{m/n}$ is a Liouville number. We need the following estimate for the length of $\alpha_{m/n}$ up to stage i .

$$l_i = \sum_{m=1}^i (n2^m)m^m < ni^i \sum_{m=1}^i 2^m = ni^i O(2^i) = nO([2i]^i) = O([2i]^i),$$

noting that n is a constant that does not depend on i .

The i^{th} convergent to $\alpha_{m/n}$ is the rational

$$\frac{\alpha_{m/n}[0 \dots l_{i-1}]}{2^{l_{i-1}}} + \frac{(0^{2^i})^{(n-m)} B(2, i)}{(2^{n2^i} - 1)2^{l_{i-1}}}.$$

The exponent of the denominator is $O(l_{i-1})$, but the distance of $\alpha_{m/n}$ from the convergent is

$$2^{-iO(l_{i-1})}.$$

This completes the proof.

7 A Conditional Construction Based on Artin's Conjecture

In this section, we give an outline of how number-theoretic properties may be employed to construct normal Liouville numbers.

Definition 4. *A number b is said to be a primitive root of a number n if the sequence $b \bmod n, b^2 \bmod n, b^3 \bmod n, \dots, b^{n-1} \bmod n$, has $n-1$ distinct elements.*

For example, 2 is a primitive root of the prime 13, but is not a primitive root of the prime 7.

Recall that the base- b expansion of any rational is eventually periodic - ignoring some finite prefix, the expansion consists of a recurring block. We mention the following observation. If b is a primitive root of p , then the fraction b/p when expressed in base b , has a recurring block of length $p-1$ - that is, the maximal length. We explain this observation as follows.

Let $1/p$ consist of repetitions of the block $b_1 \dots b_k$, each b_i a symbol in $[b]$. Then the length of the recurring block (k), is equal to the number of times the base- k expansion has to be shifted left before the fractional part represents $1/p$ again. The orbit of the number $1/m$ under the base- b left-shift is precisely the set

$$\left\{ \frac{b}{p} \bmod 1, \dots, \frac{b^k}{p} \bmod 1 \right\} = \left\{ \frac{b \bmod p}{p}, \dots, \frac{b^k \bmod p}{p} \right\}.$$

By the discussion above, the maximal k , equal to $p - 1$, is attained when b is a primitive root of p . The following lemma says that if b is a primitive root of p , the left-shifts of the base b expansion of $1/p$ show certain degree of uniformity in distribution. Thus the base- b expansion of $1/p$ is an analogue of a b -ary deBruijn sequence.

Lemma 3. *Let b be a primitive root of p . Then for each k with $b^k < p$, for each $0 \leq j < b^k$, we have the following frequency estimate:*

$$\frac{|\{1 \leq m \leq p \mid b^m p \bmod 1 \in [\frac{j}{b^k}, \frac{j+1}{b^k})\}|}{p} = \frac{1}{b^k} \pm \frac{2}{p}. \quad (5)$$

Proof. The intervals

$$[j/p, (j+1)/p), \quad 0 \leq j \leq p-1, \quad (6)$$

have the same length, and the first $p-1$ left-shifts of the base- b expansion of $\frac{1}{p}$ visit each interval exactly once. Let k be such that $b^k > p$. Since b does not divide p , no point of the form m/b^k can coincide with any of the endpoints of the intervals in (6). Hence each b -adic interval contains exactly $\lfloor b^k/p \rfloor$ many extremities in (6). Thus the number of times that the first $p-1$ left-shifts of the base- b expansion of $1/p$ visits each b -adic interval of order k , is $\lfloor \frac{p}{b^k} \rfloor \pm 2$, accounting for the fact that two intervals of the form (6) only have a partial intersection with the b -adic interval.

Thus, the frequency with which the first $p-1$ left-shifts of the b -adic expansion of $1/p$ visits any b -adic interval of order k , is $1/b^k + \frac{2}{p}$, which is the required result. \square

Consequently for each fixed k , as $p \rightarrow \infty$, the frequency in (5) tends to $1/b^k$.

We could construct a normal number along the lines of section 4 if we are assured of an infinite number of such p for each a . This is a consequence of *Artin's Conjecture*:

Conjecture 1. [1] Let b be an integer other than -1, and a perfect square. Then for any $x \in \mathbb{N}$, the number of primes for which b is a primitive root is asymptotically $C(b) \frac{x}{\log x}$, where $C(b)$ is a constant dependent on b .

In 1985, Heath-Brown [11] proved that the conjecture is true for all prime b with at most two exceptions, and for any $x \in \mathbb{N}$, the number of $b \in \mathbb{Z}$, $|a| \leq x$, for which the conjecture fails is $o((\log x)^2)$.

We need only the consequence that any such b is a primitive root of infinitely many primes, for the following construction. (The density estimates help lower bound the stage lengths, but are not strictly needed for the construction.) Let $b \geq 2$ be a number which is the primitive root of infinitely many primes $p_1 < p_2 < \dots$. For each p_i , $i \geq 1$, let BLOCK_i be the block which recurs in the base- b expansion of $\frac{1}{p_i}$. Note that $|\text{BLOCK}_i| = p_i - 1$.

Consider the number

$$x = \text{BLOCK}_1^{i_1} \text{BLOCK}_2^{i_2} \dots,$$

where i_1, i_2, \dots is a sequence of numbers such that both the following conditions hold.

1. Condition for Normality to the base- b : For each natural number n ,

$$p_n - 1 = o\left(\sum_{j=1}^k i_j\right).$$

2. Condition for x to be a Liouville number: For each number n ,

$$\left(\sum_{j=1}^{n-1} i_j(p_j - 1) + (p_n - 1)\right) n = O(i_n p_n).$$

We observe that both conditions are true for all sufficiently large numbers i_n , sufficiently rapidly growing in n , hence such a sequence of numbers $i_1 < i_2 < \dots$ exists.

The proof that the above two conditions are sufficient to ensure Liouville criterion and normality is exactly analogous to the proofs in sections 4 and 6.

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